PHYS 798C Spring 2022
Lecture 14 Summary

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I. GINZBURG LANDAU THEORY OF SUPERCONDUCTORS

BCS theory is wonderful, but we need to find new theoretical tools to attack problems that arise from defects, surfaces, interfaces with normal metals, etc. The Ginzburg-Landau (GL) theory of superconductivity is one of the most useful tools for doing quantitative calculations under interesting realistic scenarios.

BCS theory applies to homogeneous uniform superconductors, and provides insights into the microscopic pairing mechanism. However many problems of interest involve inhomogeneities such as interfaces, impurities, magnetic vortices, etc. For this we need to generalize BCS theory. One approach is to employ the Bogoliubov self-consistent field method. One can write the creation and annihilation operators in terms of real-space eigenfunctions, rather than momentum space eigenfunctions. The generalization of the Bogoliubov-Valatin transformation for the $c_{k,\sigma}$ in terms of the γ operators is:
$\Psi(\vec{r},\uparrow) = \sum_n \left[\gamma_{n,\uparrow} u_n(\vec{r}) - \gamma^+_{n,\downarrow} v^*_n(\vec{r})\right]$,
$\Psi(\vec{r},\downarrow) = \sum_n \left[\gamma_{n,\downarrow} u_n(\vec{r}) + \gamma^+_{n,\uparrow} v^*_n(\vec{r})\right]$,
where n labels the real space eigenfunctions. The first of these ($\Psi(\vec{r},\uparrow)$) is an operator that annihilates a position eigenfunction representing a particle at location $\vec{r}$ and spin $\uparrow$. Note that the u's and v's are now position dependent, and one does a functional variational calculation to minimize the Hamiltonian of the system. This will lead to the definition of a position-dependent gap function (better thought of as a *pairing potential*), determined self-consistently,
$\Delta(\vec{r}) = V\left\langle \Psi(\vec{r},\uparrow)\Psi(\vec{r},\downarrow)\right\rangle = V\sum_n v^*_n(\vec{r})u_n(\vec{r})(1-2f_n)$.

A. Ginzburg Landau Theory

Gor'kov showed that Ginzburg-Landau (GL) theory can be derived from this real-space generalized version of BCS. However we will follow the phenomenological definition of the theory.

We introduce a spatially dependent complex order parameter $\psi(\vec{r})$ which has the interpretation that the magnitude squared describes the local superconducting electron density, $|\psi(\vec{r})|^2 = n_s(\vec{r})$.
This order parameter shares many of the properties of the macroscopic quantum wavefunction that we discussed early in the semester as being responsible for the MQ properties such as flux quantization and the Josephson effect. It is *not* useful to think of the GL order parameter as a local version of the spectral gap in quasiparticle excitations.

GL proceeds by making the following postulate: ψ is "small" and varies "slowly" in space. In this case the free energy difference between the superconducting and normal states can be expanded in powers of the order parameter and its gradient as,
$f_s = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\vec{\bigtriangledown} - e^*\vec{A}\right)\psi\right|^2 + \frac{\mu_0 H^2}{2}$, where f is the free energy density (measured in J/m^3) and $\vec{A}$ is the total vector potential due to both external and self-generated fields. This expansion is expected to converge near the transition temperature where the order parameter magnitude continuously goes to zero.

B. Ginzburg Landau Temperature Dependence

In the absence of gradients and fields, the simple GL model reduces to,
$f_s = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$. To keep the order parameter finite requires that β be positive. For α positive, the minimum free energy difference occurs for $\psi = 0$. If α changes sign, there is a minimum for $\psi = \psi_\infty$ such that $|\psi_\infty|^2 = -\alpha/\beta$. The free energy density of the superconductor is lower than that of the normal

metal by an amount $f_s - f_n = -\frac{1}{2}\frac{\alpha^2}{\beta}$. We treat this as equivalent to a magnetic field energy density associated with the thermodynamic critical field H_c as,
$f_s - f_n = -\frac{1}{2}\frac{\alpha^2}{\beta} = -\frac{\mu_0 H_c^2}{2}$, so that $H_c^2 = \frac{\alpha^2}{\mu_0 \beta}$.

It is clear that α must change sign at T_c, hence we can write $\alpha(t) = \alpha'(t-1)$ where $t \equiv T/T_c$ is the reduced temperature and α' is positive. Putting this into the definition of ψ_∞ yields the temperature dependence of the order parameter near T_c:
$\psi_\infty^2 \propto n_s \propto (1-t)$, so that the order parameter falls continuously to zero at T_c linearly in temperature.

C. Ginzburg Landau Gradient Term

The gradient term is
$G = \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\vec{\bigtriangledown} - e^*\vec{A}\right)\psi\right|^2$.
where m^* and e^* are the effective mass and charge of the entities that make up the superconducting 'fluid'.
Substituting $\psi = |\psi(\vec{r})|\, e^{i\phi(\vec{r})}$ and simplifying yields,
$G = \frac{1}{2m^*}\left[\hbar^2(\bigtriangledown|\psi|)^2 + (\hbar\vec{\bigtriangledown}\phi - e^*\vec{A})^2|\psi|^2\right]$.
This shows that creating a gradient in the magnitude of the order parameter costs energy. Likewise creating a strong twist in the phase is also energetically costly. Recalling our earlier results for the MQWF, we can write the second term as $\frac{1}{2}m^* v_s^2|\psi|^2$ which can be interpreted as the kinetic energy density of the superconducting electron flow, created by a twist in phase and the electromagnetic momentum associated with the vector potential.

D. GL Effective Penetration Depth

We now define an effective magnetic penetration depth as
$\lambda_{eff}^2 \equiv \frac{m^*}{\mu_0|\psi|^2(e^*)^2}$. This differs from the London penetration depth in the substitution of the effective superconducting electron density $|\psi|^2$ for the total electron density n. As such, the effective screening length is often longer than the London penetration depth, as we shall see later. We know from above that $|\psi|^2$ is temperature dependent, hence the effective penetration depth is also temperature dependent. We also know that $|\psi_\infty|^2 \to 0$ at T_c, hence the effective penetration depth diverges at T_c.

Using the above definitions, we can now write the two phenomenological GL parameters in terms of measurable quantities as follows,
$\alpha(T) = -\frac{2\mu_0^2 e^2}{m} H_c^2(T)\lambda_{eff}^2(T)$,
$\beta(T) = \frac{4\mu_0^3 e^4}{m^2} H_c^2(T)\lambda_{eff}^4(T)$.

Using the empirical temperature dependencies $H_c \propto (1-t^2)$ and $\lambda_{eff}^{-2} \propto (1-t^4)$ yields the following temperature dependencies near T_c:
$|\psi|^2 \propto 1-t$,
$\alpha \propto -H_c^2\lambda_{eff}^2 \propto -(1-t)$, which is the expected result, but is a bit surprising since λ_{eff} diverges at T_c,
$\beta \propto H_c^2\lambda_{eff}^4 \propto \frac{1}{(1+t^2)^2} \sim$ constant near T_c.

E. GL Differential Equation

The total free energy difference
$\int(f_s - f_n)d^3r$
is a functional of the GL order parameter function $\psi(r)$ and the total vector potential $A(r)$. In the absence of fields and currents and gradients, the order parameter is equal to ψ_∞ everywhere. However, when fields and currents are present, the order parameter will become position dependent, and we want to find the resulting $\psi(\vec{r})$ that minimizes the total free energy difference. By demanding that the free energy difference is a minimum for a first order functional variation of $\psi(r)$, one can derive the GL differential equation:

$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\vec{\bigtriangledown} - e^*\vec{A}\right)^2\psi = 0.$

Note that this form implicitly assumes local electrodynamics, i.e. that there is a purely local relation between vector potential and the resulting current. Similarly for a functional variation of the vector potential ($\vec{J}_s = -\frac{\partial F_s}{\partial \vec{A}(\vec{r})}$ since $\vec{J}$ and $\vec{A}$ are conjugate variables) one finds,

$\vec{J} = \frac{e^*}{m^*}|\psi|^2\left(\hbar\vec{\bigtriangledown}\theta - e^*\vec{A}\right)$
$= e^*|\psi|^2\vec{v}_s.$

Note that the two coupled GL equations stated above have to be solved self-consistently. One can view the GL differential equation as a nonlinear Schrodinger equation with a potential of the form $V = \beta|\psi|^2$ and eigenvalue $-\alpha$. Since $\beta > 0$ the potential is repulsive, tending to spread out the distribution of $\psi(\vec{r})$. Note that superposition does not apply to solutions of this equation. The current equation is just the generalized London relation from Lecture 4, derived in the context of a macroscopic quantum wavefunction for the superconductor. It leads directly to fluxoid quantization, which we now incorporate into GL theory. The GL theory generalizes the London theory to the case of spatially varying n_s, and includes the nonlinearity that results when strong fields and current cause n_s to decrease.

F. "Derivation" of GL Theory

Starting from the BCS self-consistent gap equation one can expand the integral to third order in the limit of small gap (near T_c). Generalizing the homogeneous BCS gap $\Delta(T)$ to a spatially-dependent "pairing potential" $\Delta(\vec{r})$ and including the possibility of a gradient in the gap, as well as magnetic fields, yields this equation (Ketterson and Song, section 45);

$$\left[\left(1 - \frac{T}{T_c}\right) + \frac{1}{6}\left(\frac{\hbar v_F}{\pi k_B T_c}\right)^2 \frac{7}{8}\zeta(3)\chi(\rho)\left(\vec{\nabla} - \frac{2ie}{\hbar c}\vec{A}\right)^2 - \frac{7\zeta(3))}{8(\pi k_B T_c)^2}|\Delta(\vec{r})|^2\right]\Delta(\vec{r}) = 0.$$

This is equivalent to the GL differential equation with the assignments:

$\psi(\vec{r}) = \sqrt{\frac{7\zeta(3)n}{8(\pi k_B T_c)^2}}\Delta(\vec{r})$
$\alpha = -\frac{6\pi^2(k_B T_c)^2}{7\zeta(3)\epsilon_F}\left(1 - \frac{T}{T_c}\right)$
$\beta = \frac{12\pi^2(k_B T_c)^2}{7\zeta(3)\epsilon_F n}$

where n is the total electron number density and ϵ_F is the Fermi energy.

The pairing potential $\Delta(\vec{r})$ is no longer the gap in the excitation spectrum of the superconductor. Instead it can be thought of as a local internal field tending to produce Cooper pairing of electrons. It is also a complex function, in general.